REPORT CTSL– **25**

EFFECTS OF MAGNET NON-LINEARITIES ON BETATRON OSCILLATION FREQUENCIES FOR THE 300 BEV PROTON SYNCHROTRON

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APRIL 3, 1961

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Supported in part by the U.S. Atomic Energy Commission

Contract No. AT(11-1)-68

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I. Introduction

The purpose of this note is to study the effects on betatron frequencies -- in particular, their variation with particle energy -- due to non-linear magnet fields. The idea is to determine whether non-linearities can be introduced deliberately in such a way that ν, the number of betatron wavelengths in the circumference, is independent of particle energy for energies near the equilibrium energy. The result is that non-linearities of a magnitude such that the field index n varies by about 1 per cent per cm will suffice.

II Equations of Motion

The differential equations describing horizontal and vertical betatron oscillations are

$$\frac{d^2x}{ds^2} + \left[k^2 \left(\frac{2p_o}{p} - 1\right) + \frac{e}{p}\ \frac{dB}{dx}\right] x = - k \left(\frac{p_o}{p} - 1\right)$$

$$- \left(k^3 \frac{p_o}{p} + 2k \frac{e}{p}\ \frac{dB}{dx} + \frac{e}{2p}\ \frac{d^2B}{dx^2}\right) x^2 + \dots \qquad (1)$$

$$\frac{d^2z}{ds^2} - \frac{e}{p}\ \frac{dB}{dx}\ z = \left(2k \frac{e}{p}\ \frac{dB}{dx} + \frac{e}{p}\ \frac{d^2B}{dx^2}\right) xz + \dots\dots \qquad (2)$$

respectively. s measures distance along the equilibrium orbit, in a counter-clockwise direction when viewed from above. x and z are the departures from the equilibrium orbit, outward and downward, respectively. The magnetic field in the orbit plane is assumed to be

$$B(x) = B + x \frac{dB}{dx} + \frac{1}{2} x^2 \frac{d^2B}{dx^2} + \dots\dots$$

in a downward (+ z) direction. Note that B, dB/dx and d^2B/dx^2 are constants, measured at $x = 0$. k denotes the curvature of the equilibrium orbit, traversed by a particle of momentum p_o; p denotes the actual particle momentum. Balancing magnetic and centrifugal forces gives

$$B = \frac{p_o \, k}{e} \tag{3}$$

We shall now summarize briefly the linear theory of oscillations about the equilibrium orbit[1]; we set $p = p_o$ and drop the non-linear terms on the right of Eqs. (1) and (2). Both equations are then of the form

$$\frac{d^2 y}{ds^2} + K(s)\, y = 0 \tag{4}$$

In an alternating-gradient synchrotron K(s) will be periodic with some period L, which may be the circumference C or some fraction of C. The general solution of (4) may be written

$$y(s) = A \; w(s) \cos\left[\psi(s) + \delta\right] \tag{5}$$

where A and δ are arbitrary constants, $w(s + L) = w(s)$, $\psi(s + L) = \psi(s) + \mu$ and

$$\frac{d\psi}{ds} = \frac{1}{w^2} \tag{6}$$

μ is the betatron phase shift in L. The quantity $w^2(s)$ is generally called $\beta(s)$; according to Eq. (6), it represents the local betatron (reduced) wavelength.

The number of betatron oscillations in the machine circumference is

$$\nu = \frac{N\mu}{2\pi} \tag{7}$$

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1) A good review of the theory is given by Courant and Snyder, Annals of Physics 3, 1 (1958).

where N is the number of identical sections; $C = NL$.

Now consider a small perturbation $\delta(s)$ in $K(s)$; that is, consider instead of Eq. (4) the equation

$$\frac{d^2y}{ds^2} + \left[K(s) + \delta(s) \right] y = 0 \tag{8}$$

It can be shown[1,2] that, to first order in δ, the change in ν is given by

$$\Delta\nu = \frac{1}{4\pi} \oint \delta(s)\ \beta(s)\ ds$$

where $\oint$ denotes an integral around the entire machine circumference.

III Effects of Synchrotron Oscillations

Suppose now that the particle momentum p differs slightly from the equilibrium momentum p_o. The equilibrium orbit is then shifted outward by an amount $X(s)$. We now write

$$x(s) = X(s) + \xi(s) \tag{9}$$

where ξ describes the rapid horizontal betatron oscillations about the slowly varying orbit X. Substituting Eq. (9) into (1) and (2), we obtain the differential equations

$$\frac{d^2\xi}{ds^2} + \left[k^2 \left(1 - 2\frac{\Delta p}{p_o}\right) + \frac{e}{p}\frac{dB}{dx} + 2X\left(k^3 \frac{p_o}{p} + 2k\frac{e}{p}\frac{dB}{dx} + \frac{e}{2p}\frac{d^2B}{dx^2}\right) \right] \xi = 0 \tag{10}$$

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2) L. S. Smith, Internal Report LS-4, Brookhaven National Laboratory.

$$\frac{d^2z}{ds^2} + \left[- \frac{e}{p} \frac{dB}{dx} - X \left(2k \frac{e}{p} \frac{dB}{dx} + \frac{e}{p} \frac{d^2B}{dx^2}\right) \right] z = 0 \tag{11}$$

where we have linearized in the small quantities ξ and z.

Equations (10) and (11) are now of the form (8). The unperturbed K(s) for $p = p_o$ are

$$K = \begin{cases} k^2 + \dfrac{e}{p_o} \dfrac{dB}{dx} & \text{(horizontal oscillations)} \\ \\ - \dfrac{e}{p_o} \dfrac{dB}{dx} & \text{(vertical oscillations)} \end{cases}$$

so that the horizontal and vertical perturbations $\delta(s)$ are

$$\delta_h = - 2k^2 \frac{\Delta p}{p_o} - \frac{\Delta p}{p_o} \frac{e}{p_o} \frac{dB}{dx} + 2X \left(k^3 + 2k \frac{e}{p_o} \frac{dB}{dx} + \frac{e}{2p_o} \frac{d^2B}{dx^2}\right)$$

$$\delta_v = \frac{\Delta p}{p_o} \frac{e}{p_o} \frac{dB}{dx} - X \left(2k \frac{e}{p_o} \frac{dB}{dx} + \frac{e}{p_o} \frac{d^2B}{dx^2}\right)$$

It is conventional to define $n = \frac{-1}{kB} \frac{dB}{dx} = - \frac{e}{p_o k^2} \frac{dB}{dx}$. This gives

$$\delta_h = \frac{\Delta p}{p_o} (n - 2)k^2 + 2X \left[k^3(1 - 2n) + \frac{e}{2p_o} \frac{d^2B}{dx^2} \right] \tag{12}$$

$$\delta_v = - n\, k^2 \frac{\Delta p}{p_o} - X \left(- 2n\, k^3 + \frac{e}{p_o} \frac{d^2B}{dx^2}\right) \tag{13}$$

Two simplifications can now be made. In the first place, $|n| \gg 1$ (for the 300 Bev machine, $|n| \sim 10^4$). In the second place, we can use the large value of ν ($\sim$ 40) as follows: Smith[2)] gives the approximation

$$X(s) \approx \frac{\Delta p}{p_o} \left\langle k\, \beta^{3/2} \right\rangle \beta^{1/2}(s) \tag{14}$$

where $\langle\ \rangle$ denotes an average over s around the circumference. Therefore, omitting factors of order one,

$$X \sim \frac{\Delta p}{p_o}\, k\, \beta^2$$

From Eqs. (6) and (7),

$$\beta \sim \frac{C}{2\pi\nu} \quad \frac{R}{\nu}$$

where R is the machine radius $\frac{C}{2\pi}$. Therefore

$$X \sim \frac{\Delta p}{p_o}\, \frac{kR^2}{\nu^2} \sim \frac{\Delta p}{p_o}\, \frac{R}{\nu^2} \sim \frac{\Delta p}{p_o}\, \frac{1}{k\nu^2} \tag{15}$$

The large value of ν, coupled with (15), enables us to omit the $X\, k^3$ terms in Eqs. (12) and (13). Thus, (12) and (13) become

$$\delta_h = n\, k^2 \frac{\Delta p}{p_o} + \frac{k}{B} \frac{d^2 B}{dx^2} X$$

$$\delta_v = -n\, k^2 \frac{\Delta p}{p_o} - \frac{k}{B} \frac{d^2 B}{dx^2} X$$

We made use of Eq. (3) to simplify the last term in each.

The change in effective gradient length with orbit position produces a significant shift in ν for the Brookhaven AGS, and we shall therefore include this effect. According to Smith[2)], this contributes to $\begin{pmatrix}\delta_h \\ \delta_v\end{pmatrix}$ a term

$$\mp \frac{1}{2} |n| k^2 \left| \frac{dLg}{dX} \right| X \delta(s)_{end}$$

where $\delta(s)_{end}$ consists of a delta function at each end of every magnet.

We may summarize by observing that there are three perturbing terms in δ_h (and δ_v), due to momentum error, magnet non-linearity, and variation in gradient length. Also, note that $\delta_h = -\delta_v$ in this approximation.

IV Numerical Estimates

Since no definite magnet lattice has been arrived at for the 300 Bev machine, we have adopted some tentative and approximate numbers. We shall assume

$$R = 1300 \text{ m}$$

$$k = (1000 \text{ m})^{-1}$$

$$n = \pm 10^4$$

$$N = 30$$

$$\nu = 40$$

Each of the 30 "superperiods" contains 32 magnets, 16 radially focussing (F) and 16 radially defocussing (D). Each magnet is 7 meters long. The phase shift per unit length is

$$\frac{2\pi\nu}{C} = \frac{\nu}{R} \approx 3 \times 10^{-2} \text{ m}^{-1}$$

so that $\beta \approx 33$ m. We shall assume that β oscillates between 20 m and 50 m; more precisely, we assume

$$\beta_h = \begin{cases} 20 \text{ m} & \text{in} \quad \text{D magnets} \quad (n = +10^4) \\ \\ 50 \text{ m} & \text{in} \quad \text{F magnets} \quad (n = -10^4) \end{cases}$$

For vertical oscillations the situation is reversed; β_v = 20 m in F magnets and 50 m in D magnets.

For X(s) we shall make the approximation (14). For our assumed lattice

$$\left\langle k\, \beta^{3/2} \right\rangle \approx \frac{1}{1.3} \cdot \frac{1}{1000} \cdot \frac{20^{3/2} + 50^{3/2}}{2}$$

$$\approx 0.17 \text{ m}^{1/2}$$

Thus

$$X(s) \approx \begin{cases} 0.76 \dfrac{\Delta p}{p_o} \text{ m} & \text{in} \quad \text{D magnets} \\ \\ 1.20 \dfrac{\Delta p}{p_o} \text{ m} & \text{in} \quad \text{F magnets} \end{cases}$$

We shall now separately estimate the three effects referred to in the previous section:

<u>(a) Variation in gradient length</u> -

$$\Delta\nu_h = \frac{1}{4\pi} \oint \beta_h(s)\, \delta_h(s)\, ds$$

$$= \frac{-1}{8\pi} \left| n \right| k^2 \left| \frac{dIg}{dX} \right| \sum_{\text{ends}} X(s)\, \beta_h(s)$$

A reasonable estimate for $\left| \dfrac{dIg}{dX} \right|$ seems to be 0.5. Also

$$\sum_{\text{ends}} X(s)\ \beta_h(s) \approx 960 \left[(0.76) \times (20) + (1.20) \times (50)\right] \frac{\Delta p}{p_o}$$

$$\approx 7.2 \times 10^4 \ \frac{\Delta p}{p_o}\ m^2$$

Therefore

$$\Delta\nu_h \approx \frac{-1}{8\pi}\ \frac{10^4}{10^6}\ (0.5)\ (7.2 \times 10^4)\ \frac{\Delta p}{p_o}$$

$$\approx -\ 14.3\ \frac{\Delta p}{p_o}$$

For vertical oscillations

$$\Delta\nu_v = \frac{+1}{8\pi}\ |n|\ k^2 \left|\frac{dIg}{dX}\right| \sum_{\text{ends}} X(s)\ \beta_v(s)$$

Estimating as before

$$\sum_{\text{ends}} X(s)\ \beta_v(s) \approx 960 \left[(0.76) \times (50) + (1.20) \times (20)\right] \frac{\Delta p}{p_o}$$

$$\approx 6.0 \times 10^4 \ \frac{\Delta p}{p_o}\ m^2$$

and

$$\Delta\nu_v \approx +\ 11.9\ \frac{\Delta p}{p_o}$$

(b) Momentum error -

$$\Delta\nu_h = \frac{1}{4\pi}\ \frac{\Delta p}{p_o} \oint n\ k^2\ \beta_h\ ds$$

$$\oint n\, k^2\, \beta_h\, ds \approx \frac{10^4}{10^6}\,(480)(7)(20 - 50)$$

$$\approx -1010$$

so that

$$\underline{\underline{\Delta\nu_h \approx -80\,\frac{\Delta p}{p_o}}}$$

A similar calculation gives

$$\underline{\underline{\Delta\nu_v \approx -80\,\frac{\Delta p}{p_o}}}$$

<u>(c) Magnet non-linearity</u> -

$$\Delta\nu_h = \frac{1}{4\pi}\oint \frac{k}{B}\,\frac{d^2B}{dx^2}\,X\,\beta_h\, ds$$

Let us assume

$$\frac{1}{k^2B}\,\frac{d^2B}{dx^2} = \begin{cases} A & \text{in D magnets} \\ B & \text{in F magnets} \end{cases}$$

Then

$$\Delta\nu_h \approx \frac{(480)(7)}{4\pi \times 10^9}\,\frac{\Delta p}{p_o}\left[(0.76)(20)A + (1.20)(50)B\right]$$

$$\underline{\underline{\Delta\nu_h \approx (4.1\,a + 16.1\,b)\,\frac{\Delta p}{p_o}}}$$

where we have set

$$a = 10^{-6} A, \qquad b = 10^{-6} B$$

Similarly

$$\Delta\nu_v \approx - \frac{(480)(7)}{4\pi \times 10^9} \frac{\Delta p}{p_o} \left[(0.76)(50)A + (1.20)(20)B\right]$$

$$\underline{\underline{\Delta\nu_v \approx (- 10.2\ a - 6.4\ b) \frac{\Delta p}{p_o}}}$$

V Optimum Amount of Non-linearity

Collecting the numerical results of the previous section, we have

$$\Delta\nu_h = (- 14.3 - 80 + 4.1\ a + 16.1\ b) \frac{\Delta p}{p_o}$$

$$\Delta\nu_v = (11.9 - 80 - 10.2\ a - 6.4\ b) \frac{\Delta p}{p_o}$$

We have exhibited explicitly the variation in gradient length effect, the momentum error effect, and the magnet non-linearity effects, in that order. Note that here, in contrast to the Brookhaven AGS, the first effect is quite small compared to the second.

If we choose

$$\begin{aligned} a &= 0.7 - 13.0 = - 12.3 \\ b &= 0.7 + 8.3 \ = + 9.0 \end{aligned} \qquad (16)$$

the effects cancel and ν_h and ν_v are independent of $\frac{\Delta p}{p_o}$. In Eq. (16) the gradient length and momentum error contributions are presented separately, in that order.

Note that

$$\frac{\Delta n}{\Delta x} = \frac{-1}{kB} \frac{d^2B}{dx^2}$$

$$= - kA \quad \text{or} \quad -kB$$

$$= - 10^3 a \text{ or } - 10^3 b$$

Thus the result is that

$$n = \begin{cases} 10{,}000 + 123\, x_{cm} & \text{in} \quad \text{D magnets} \\ \\ -\ 10{,}000 - 90\, x_{cm} & \text{in} \quad \text{F magnets} \end{cases}$$

where x_{cm} is simply x expressed in centimeters.

These non-linearities are clearly achievable; the accuracy with which they can be maintained would appear to be a more serious problem.